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The sign regularity of the auxiliary family $g_i(x) = x^{\alpha_i}(-\ln x)^{\beta_i}$ in convergence acceleration processes using the E-algorithm

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Abstract

The E-algorithm can be successfully applied to accelerate the convergence of the solution of some numerical differential equations provided the respective error expansion satisfies a certain asymptotic determinantal condition. In this work it is proved that such condition is fulfilled for any error expansion in which formation enter the auxiliary family $h_n^{\alpha_i}(-\ln h_n)^{\beta_i}$ (where h_n designates the stepsize of the discretization method).

Keywords: Error expansion; E-algorithm; Convergence acceleration; Sign regularity

1. Introduction

Differential equations which have numerical solutions converging very slowly when the stepsize tends to zero occur quite frequently. Sometimes it is possible to use extrapolation methods such as the well known Richardson extrapolation to accelerate the convergence of approximate solutions (see [6] and references therein). Moreover, for some problems, better numerical results can be achieved by means of more recent and efficient extrapolation methods such as the E-algorithm [1–4]. This last method is particularly useful if an asymptotic expansion of the error is a priori known, as it is the case of the numerical differential problems treated in [6, 7]. The successful numerical results obtained in the last of these works – where a dramatic improvement of convergence by using the E-algorithm has been achieved – were not theoretically explained with respect to the convergence and acceleration for all examples given there. In particular, Example 2 in [7] consisted on the computation of $u(1/2)$ with $f(x) = \ln(x)/x$ for the boundary value problem

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}u}{\mathrm{d}x}\right) - qu = -f, \tag{1.1}$$

$$u(0) = u(1) = 0, \tag{1.2}$$

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where

$$p(x) = 1 + |x - \tfrac{1}{2}|, \tag{1.3}$$

and

$$q(x) = \begin{cases} 0, & \text{if } x < \frac{1}{2}, \\ \frac{1}{2}, & \text{if } x = \frac{1}{2}, \\ 1, & \text{if } x > \frac{1}{2}. \end{cases} \tag{1.4}$$

The discretization of this problem was made using the finite-difference scheme

$$\frac{1}{h^2}(\bar{p}_{i+1}(u_{i+1}^h - u_i^h) - \bar{p}_i(u_i^h - u_{i-1}^h)) - q_i u_i^h = -f_i, \quad i = 1, 2, \ldots, n-1, \tag{1.5}$$

$$u_0^h = u_n^h = 0, \tag{1.6}$$

where $\bar{p}_i = p(x_i - h/2)$, $q_i = q(x_i)$, $f_i = f(x_i)$, $x_i = ih$ and h is the stepsize of the finite-difference method. In order to be able to extrapolate the values computed with (1.5)–(1.6) we considered a sequence of stepsizes $h_n = \frac{1}{2}^{n+1}$, $n = 0, 1 \ldots$. For each approximate solution u^{h_n} the following asymptotic error expansion holds (see Section 3–3 in [7]):

$$u^{h_n} = u + A_1 h_n(-\ln h_n) + B_1 h_n + A_2 h_n^2(-\ln h_n) + B_2 h_n^2 + A_3 h_n^3(-\ln h_n) + B_3 h_n^3 + \cdots. \tag{1.7}$$

Let $u(x)$ be the solution of (1.1)–(1.2). Example 4 in [7] consisted in the computation of the integral $\Phi = \int_0^1 f(x)u(x)\,dx$ with $f(x) = \ln x/x$, the respective asymptotic error expansion being

$$\begin{aligned}\Phi^{h_n} = \Phi &+ A_1 h_n(-\ln h_n)^3 + B_1 h_n(-\ln h_n)^2 + C_1 h_n(-\ln h_n) + D_1 h_n \\ &+ A_2 h_n^2(-\ln h_n)^3 + B_2 h_n^2(-\ln h_n)^2 + C_2 h_n^2(-\ln h_n) + D_2 h_n^2 \\ &+ A_3 h_n^3(-\ln h_n)^3 + B_3 h_n^3(-\ln h_n)^2 + C_3 h_n^3(-\ln h_n) + D_3 h_n^3 + \cdots.\end{aligned} \tag{1.8}$$

(The minus sign before $\ln h_n$ in (1.7) and (1.8) enables exponentiation with positive base). Now let $(x_n) \xrightarrow[n\to\infty]{} 0$ be any auxiliary sequence (in the above examples the sequence of the stepsizes) and (S_n) be a sequence converging to a limit S. If an asymptotic error expansion is known, like

$$S_n = S + a_1 g_1(x_n) + a_2 g_2(x_n) + \cdots + a_k g_k(x_n) + \cdots, \quad n = 0, 1, \ldots \tag{1.9}$$

we can compute extrapolated values T_n by means of the E-algorithm and hope that the sequence (T_n) converges faster than (S_n) (that is $\lim(T_n - S)/(S_n - S) = 0$). Motivated by the examples referred above, our purpose is to show that the family $g_i(x) = x^{\alpha_i}(-\ln x)^{\beta_i}$ $i = 1, 2, \ldots, x \to 0$ (see (2.7) below) arising in the error expansions of certain boundary value problems (see [6, 7]) do satisfy a determinantal condition, which guarantees (by a theorem of Matos and Prévost [8]) the convergence and convergence acceleration of the E-algorithm. Such condition translates the sign regularity nature of the auxiliary family $g_i(x) = x^{\alpha_i}(-\ln x)^{\beta_i}$ which means that certain determinants, defined in terms of the g_i's functions, are not negative.

In Section 2 we define a set of functions, depending on the g_i's and denote by $g_i^{[j]}(x)$, and we conclude that these functions are positive on (0, 1) or on a smaller interval. Finally, in Section 3, we analyse the sign of certain Wronskians in connexion with the sign of the $g_i^{[j]}$'s. This will enable us to prove at last the sign regularity of the auxiliary family $x^{\alpha_i}(-\ln x)^{\beta_i}$.

2. Sign analysis

If the truncation error of a sequence (S_n) has an asymptotic expansion of the form $S_n - S = a_1 g_1(n) + a_2 g_2(n) + \cdots$, where the a_i's are constants independent of n, and the $g_i(n)$'s are an asymptotic sequence as $n \to \infty$ — that is $g_i(n) \xrightarrow[n\to\infty]{} 0$ and $g_{i+1}(n) = o(g_i(n))$ — the following Theorem 1 due to Matos and Prévost [8], [3, Theorem 2.11, p. 69] gives a sufficient asymptotic determinantal condition ensuring the convergence acceleration of the E-algorithm [1, 2, 4]:

Theorem 1 (Matos and Prévost [8, Theorem 3, p. 398]). *Let $S_n - S = a_1 g_1(n) + a_2 g_2(n) + \cdots$ such that*

$$\text{(H1)} \qquad [g_0(n) \equiv 1]\, \frac{g_{i+1}(n)}{g_i(n)} \xrightarrow[n\to\infty]{} 0, \quad i = 0, 1, 2, \ldots \tag{2.1}$$

$$\text{(H2)} \qquad \forall\, i \in \mathbb{N},\ \forall\, m \in \mathbb{N}\ \exists\, N \in \mathbb{N}\colon n \geqslant N \Rightarrow \begin{vmatrix} g_{i+m}(n) & g_{i+m-1}(n) & g_i(n) \\ \vdots & \vdots & \vdots \\ g_{i+m}(n+m) & g_{i+m-1}(n+m) & g_i(n+m) \end{vmatrix} \geqslant 0 \tag{2.2}$$

then

$$\frac{E_{k+1}^{(n)} - S}{E_k^{(n)} - S} \xrightarrow[n\to\infty]{} 0, \quad k = 0, 1, \ldots, \tag{2.3}$$

where the $E_k^{(n)}$ are computed by means of the formulas for the E-algorithm [2]:
Initializations:

$$E_0^{(n)} = S_n, \quad n = 0, 1, \ldots,$$
$$g_{0,i}^{(n)} = g_i(n), \quad i = 1, 2, \ldots \quad n = 0, \ldots. \tag{2.4}$$

For $k = 1, 2, \ldots$; for $n = 0, 1, \ldots$

$$E_k^{(n)} = E_{k-1}^{(n)} + g_{k-1,k}^{(n)} \frac{E_{k-1}^{(n)} - E_{k-1}^{(n+1)}}{g_{k-1,k}^{(n+1)} - g_{k-1,k}^{(n)}},$$
$$g_{k,i}^{(n)} = g_{k-1,i}^{(n)} + g_{k-1,k}^{(n)} \frac{g_{k-1,i}^{(n)} - g_{k-1,i}^{(n+1)}}{g_{k-1,k}^{(n+1)} - g_{k-1,k}^{(n)}}, \quad i = k+1, k+2, \ldots. \tag{2.5}$$

and are usually placed in an double-entry array, the so-called E-array as follows:

$$\begin{array}{lllll} E_0^0 = S_0 & & & & \\ E_0^1 = S_1 & E_1^0 & & & \\ E_0^2 = S_2 & E_1^1 & E_2^0 & & \\ E_0^3 = S_3 & E_1^2 & E_2^1 & E_3^0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \tag{2.6}$$

We would like to show that the determinantal condition (2.2) is satisfied (with strict inequality) for a class of auxiliary functions $g_i(x)$ occurring in the numerical solution of boundary value problems treated in [6, 7] which guarantees that the E-algorithm accelerates convergence. Those auxiliary functions belong to the following family:

$$g_i(x) = x^{\alpha_i}(-\ln x)^{\beta_i}, \quad i = 1, 2, \ldots, \quad \alpha_i, \beta_i \in \mathbb{R}^+ \quad x \in (0, 1), \ x \to 0. \tag{2.7}$$

We are only concerned with the cases for which $g_{i+1}(x) = o(g_i(x))$:

(i) $\quad g_i(x) = x^{\alpha}(-\ln x)^{\beta_i}, \quad \alpha \in \mathbb{R}^+, \ \beta_1 > \beta_2 > \cdots > \beta_m > \cdots > 0, \qquad (2.8)$

(ii) $\quad g_i(x) = x^{\alpha_i}(-\ln x)^{\beta_i}, \quad 0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < \cdots, \ \beta_1 > \beta_2 > \cdots > \beta_m > \cdots > 0, \qquad (2.9)$

(iii) $\quad g_i(x) = x^{\alpha_i}, \quad 0 < \alpha_1 < \cdots < \alpha_m < \cdots$ or $g_i(x) = (-\ln x)^{\beta_i}, \ \beta_1 > \beta_2 > \cdots > 0. \qquad (2.10)$

(iv) There is a certain number, say p, of equal α_i's with the corresponding β_j's in decreasing order. This pattern is repeated for the same β's but increasing the α's, i.e.

$$\left\{\begin{matrix} \alpha_1 = \alpha_2 = \cdots = \alpha_p \\ \beta_1 > \beta_2 > \cdots > \beta_p \end{matrix}\right\}, \ \alpha_{p+1} > \alpha_p, \ \left\{\begin{matrix} \alpha_{p+1} = \alpha_{p+2} = \cdots = \alpha_{2p} \\ \beta_1 > \beta_2 > \cdots > \beta_p \end{matrix}\right\}, \ \alpha_{2p+1} > \alpha_{2p}, \ldots. \tag{2.11}$$

Definition 2. Let $g_i^{[0]}(x) = g_i(x)$ and denote by $g_i^{[j]}(x)$ the function that can be obtained recursively from $g_i^{[0]}(x)$ by the following formula:

$$\forall j \geqslant 1, \quad \forall x \in (0, 1) \ g_i^{[j]}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g_{i+1}^{[j-1]}(x)}{g_1^{[j-1]}(x)}\right). \tag{2.12}$$

The sign of $g_i^{[j]}(x)$ is studied for (i) and (iii) in Lemma 3, and in Lemma 6 for the case (ii) with mild supplementary restrictions on the α's and β's. The analysis of such sign in case (iv) is closely related with the previous cases and will be considered using some of the examples given in [6, 7] as particular cases. The difficulty here is that the sign analysis for a particular example can be very hard due to the enormous complication of the respective $g_i^{[j]}(x)$ expressions for high j. Nevertheless it is quite remarkable that if we associate a certain number of consecutive g_i's (for instance, consider two consecutive g_i's as follows and let $g_i^{[0]}(x) = g_i(x) + g_{i+1}(x)$) the resulting new $g_i^{[j]}$'s sign can be computed quite easily as it is shown at the end of this section.

Lemma 3. *Let $g_i(x)$ be the sequence given by (2.8). Then $\forall j \geqslant 0$ $[\prod_{l=1}^{0} \equiv 1]$*

$$g_i^{[j]}(x) = -\frac{\prod_{l=1}^{j}(\beta_{i+j}-\beta_l)}{\prod_{l=1}^{j-1}(\beta_j-\beta_l)} \cdot \frac{1}{x(-\ln x)^{1+\beta_j-\beta_{i+j}}}, \quad i = 1,2,\ldots \tag{2.13}$$

and

$$\operatorname{sign} g_1^{[j]}(x) = +1 \quad j = 0,1,\ldots, \ \forall \alpha > 0, \ \forall x \in (0,1). \tag{2.14}$$

Proof. We shall use mathematical induction over j. For $j=0$ $g_i^{[0]}(x) = g_i(x) = x^{\alpha}(-\ln x)^{\beta_i}$, $i = 1,2,\ldots$ so $\operatorname{sign} g_i^{[0]}(x) = +1$. In particular, $\operatorname{sign} g_1^{[0]}(x) = +1$.

For $j = 1$

$$g_i^{[1]}(x) = \frac{\mathrm{d}}{\mathrm{d}x}((-\ln x)^{\beta_{i+1}-\beta_1}) = -\frac{\beta_{i+1}-\beta_1}{x(-\ln x)^{1+\beta_1-\beta_{i+1}}}, \quad i = 1,2,\ldots$$

and $\operatorname{sign} g_i^{[1]}(x) = +1$, $i = 1,2,\ldots$ since $\beta_{i+1} < \beta_1$. The induction hypothesis is

$$g_i^{[j]}(x) = -\frac{\prod_{l=1}^{j}(\beta_{i+j}-\beta_l)}{\prod_{l=1}^{j-1}(\beta_j-\beta_l)} \cdot \frac{1}{x(-\ln x)^{1+\beta_j-\beta_{i+j}}}, \quad i = 1,2,\ldots, \qquad \operatorname{sign} g_i^{[j]}(x) = +1 \quad \forall j \geqslant 1. \tag{2.15}$$

Then using Definition 2 we have $g_i^{[j+1]}$

$$\begin{aligned} g_i^{[j+1]}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{\prod_{l=1}^{j}(\beta_{i+j+1}-\beta_l)}{\prod_{l=1}^{j}(\beta_{j+1}-\beta_l)} \cdot (-\ln x)^{\beta_{i+j+1}-\beta_{j+1}}\right\} \\ &= -(\beta_{i+j+1}-\beta_{j+1}) \cdot \frac{\prod_{l=1}^{j}(\beta_{i+j+1}-\beta_l)}{\prod_{l=1}^{j}(\beta_{j+1}-\beta_l)} \cdot \frac{1}{x(-\ln x)^{1+\beta_{j+1}-\beta_{i+j+1}}} \end{aligned} \tag{2.16}$$

and therefore (2.15) holds for any non-negative integer j since $\beta_{i+j+1} < \beta_{j+1}$ by hypothesis.

In the case of (iii) we are led to

$$g_i^{[j]}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{\prod_{l=1}^{j-1}(\alpha_{i+j}-\alpha_l)x^{\alpha_{i+j}-\alpha_j}}{\prod_{l=1}^{j-1}(\alpha_j-\alpha_l)}\right\} = \frac{\prod_{l=1}^{j-1}(\alpha_{i+j}-\alpha_l)}{\prod_{l=1}^{j-1}(\alpha_j-\alpha_l)} x^{\alpha_{i+j}-\alpha_j-1}, \quad i = 1,2,\ldots \tag{2.17}$$

$\forall j \geqslant 1$ and then to the same sign as in (2.14).

For $g_i(x) = (-\ln x)^{\beta_i}$ the expression of $g_i^{[j]}$ is

$$g_i^{[j]}(x) = -\frac{\beta_{i+1}-\beta_1}{x(-\ln x)^{\beta_1-\beta_{i+1}}}$$

thus

$$\operatorname{sign} g_i^{[j]}(x) = +1 \quad \forall x \in (0,1). \qquad \square$$

Lemma 4. *Let*

$$\begin{aligned} C_i^{[1]} &= 1, \\ C_i^{[j+1]} &= \frac{(a_{ij}r + b_{ij}x)C_{i+1}^{[j]} + rx(C_{i+1}^{[j]})'}{(a_j r + b_j x)C_1^{[j]} + rx(C_1^{[j]})'}, \quad i,j = 1,2,\ldots, \end{aligned} \tag{2.18}$$

where $x \in (0,1)$, $r = -\ln x > 0$,

$$(C_{i+1}^{[j+1]})' = \frac{\mathrm{d}}{\mathrm{d}x}(C_{i+1}^{[j+1]})$$

and a_{ij}, b_{ij}, a_j, b_j *are quantities depending on the* α*'s and* β*'s as in* (2.9) *satisfying the following inequalities*:

$$\begin{aligned} a_{ij} &= a_{i+j+1} - \alpha_j > 0, \\ b_{ij} &= \beta_{i+j+1} - \beta_j < 0, \\ a_j &= \alpha_{j+1} - \alpha_j > 1, \\ b_j &= \beta_{j+1} - \beta_j < 0. \end{aligned} \tag{2.19}$$

Suppose further that

$$a_j b_{ij} > a_{ij} b_j, \quad i,j = 1,2,\ldots \tag{2.20}$$

Then, there exists $\delta \in \mathbb{R}^+$ *such that* $(C_i^{[j+1]})'$ *is bounded in* $(0,\delta] \subseteq (0,1)$, $j = 0,1,\ldots$.

Proof. First, let us prove that the following real function f defined in $(0,1)$ has bounded derivative in a subinterval of $(0,1)$:

$$f(x) = \frac{ar + bx}{\tilde{a}r + \tilde{b}x} \quad \text{where } \tilde{a} > 1,\ a, r = -\ln x \in \mathbb{R}^+ \text{ and } b, \tilde{b} \in \mathbb{R}^-. \tag{2.21}$$

We have

$$\left|\frac{\mathrm{d}f}{\mathrm{d}x}(x)\right| = \left|\frac{(\tilde{a}b - a\tilde{b})}{(\tilde{a}r + \tilde{b}x)^2}\right| \leqslant \frac{|\tilde{a}b - a\tilde{b}|}{\tilde{a}r + \tilde{b}x}(r+1) \leqslant \frac{|\tilde{a}b - a\tilde{b}|}{\tilde{a}r + \tilde{b}}(r+1). \tag{2.22}$$

As $x \in (0,1)$, $\tilde{a} > 0$, and $\tilde{b} < 0$, we can compute ε_1 $0 < \varepsilon_1 < 1$ such that for all x in $(0, \varepsilon_1]$ the denominator of f in (2.21) is greater than 1. This justifies the first inequality in (2.22), while the last one is valid for any $x \in (0, \varepsilon_1]$ since $\tilde{b} < 0$. Consider now $\varepsilon_2 = e^{(\tilde{b}-1)/(\tilde{a}-1)}$ then for any $x \in (0, \varepsilon_2]$ $\tilde{a}r + \tilde{b} \geqslant r + 1$. Thus, for $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$

$$\left|\frac{\mathrm{d}f}{\mathrm{d}x}(x)\right| \leqslant |\tilde{a}b - a\tilde{b}| < +\infty \quad \forall\, x \in (0, \varepsilon] \subseteq (0,1), \tag{2.23}$$

that is, f' is bounded in $(0, \varepsilon]$.

Remark 5. Hereafter when we say "*sufficiently small x*" we mean that it is possible to compute an $\varepsilon > 0$ and set the interval $(0, \varepsilon]$ in which certain property holds for all x.

With respect to f defined in (2.21), for *sufficiently small x*

$$f(x) > 0, \tag{2.24}$$

$$\operatorname{sign} f'(x) = \operatorname{sign}(\tilde{a}b - a\tilde{b}) \tag{2.25}$$

(the condition (2.24) follows from (2.21) known that sign $f(x) = \operatorname{sign}(a/\tilde{a}) = +1$ and the condition (2.25) from the expression of f' in (2.22)).

Proof of Lemma 4 (conclusion). Using induction on j, let us prove now Lemma 4. For $j = 0$ $(C_i^{[1]})' = 0$ due to (2.18), thus the conclusion is trivial.

If

$$j = 1, \quad C_i^{[2]} = \frac{a_{i1}r + b_{i1}x}{a_1 r + b_1 x}, \quad i = 1, 2, \ldots, \tag{2.26}$$

where, as in (2.19),

$$\begin{aligned} a_{i1} &= \alpha_{i+2} - \alpha_1 > 0, \\ b_{i1} &= \beta_{i+2} - \beta_1 < 0, \\ a_1 &= \alpha_2 - \alpha_1 > 1, \quad i = 1, 2, \ldots, \\ b_1 &= \beta_2 - \beta_1 < 0, \end{aligned} \tag{2.27}$$

$$a_1 b_{i1} > a_{i1} b_1, \quad i = 1, 2, \ldots \text{ (as in (2.20))}. \tag{2.28}$$

Taking $f(x) = C_i^{[2]}$ as in (2.21), we conclude by (2.23) that for *sufficiently small* x in $(0, 1)$ $(C_i^{[2]})'$ is bounded. Moreover by (2.24), $C_i^{[2]} > 0$ and due to (2.25) and (2.28) $(C_i^{[2]})' > 0$ in a certain interval contained in $(0, 1)$.

Consider now any $j \geqslant 1$ and $C_i^{[j+1]}$ given as in (2.18) under the conditions (2.19) and (2.20). $C_i^{[j+1]}$ can be rewritten as

$$C_i^{[j+1]} = \frac{\gamma_{ij} + \omega_{ij}x}{\gamma_j r + \omega_j x}, \quad i = 1, 2, \ldots, \tag{2.29}$$

where

$$\begin{aligned} \gamma_{ij} &= a_{ij} C_{i+1}^{[j]} + x(C_{i+1}^{j})', \\ \omega_{ij} &= b_{ij} C_{i+1}^{[j]}, \quad i = 1, 2, \ldots, \\ \gamma_j &= a_j C_1^{[j]} + x(C_1^{j})', \\ \omega_j &= b_j C_1^{[j]}. \end{aligned} \tag{2.30}$$

We are also assuming that for *sufficiently small* $x \in (0, 1)$ $C_{i+1}^{j}, (C_{i+1}^{j})', C_1^{[j]}$ and $(C_1^{[j]})'$ are positive. So, from (2.30) $\gamma_{ij} > 0$, $\omega_{ij} < 0$, $\gamma_j > 1$ (since $a_j > 1$ by hypothesis and $C_1^{[j]} \simeq (\alpha_{j+1} - \alpha_j)/(\alpha_j - \alpha_{j-1}) > 1$ for *sufficiently small* x) and $\omega_j < 0$. Hence, taking in (2.21) $f(x) = C_i^{[j+1]}$, $|(C_i^{[j+1]})'| < +\infty$, for *sufficiently small* x. $\square$

Lemma 6. *Let* $g_i(x) = g_i^{[0]}(x) = x^{\alpha_i}(-\ln x)^{\beta_i}(x)$ $i = 1, 2, \ldots$ *as in* (2.9) *such that*

$$\alpha_{j+1} - \alpha_j > 1 \quad \text{and} \quad (\alpha_{j+1} - \alpha_j)(\beta_{i+j+1} - \beta_j) > (\alpha_{i+j+1} - \alpha_j)(\beta_{j+1} - \beta_j) \qquad i, j = 1, 2, \ldots \tag{2.31}$$

Then, for sufficiently small $x \in (0,1)$

$$\operatorname{sign} g_1^{[1]} = \operatorname{sign}(\alpha_2 - \alpha_1), \tag{2.32}$$

$$\operatorname{sign} g_1^{[j]} = \operatorname{sign} \frac{\prod_1^{l-j}(\alpha_{j+1} - \alpha_l)}{\prod_1^{l=j-1}(\alpha_j - \alpha_l)}, \quad j = 2,3,\ldots. \tag{2.33}$$

Corollary 7. *If* $x \in (0,1)$ *is sufficiently small and* $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < \cdots$ $\beta_i \neq \beta_j$ $(i \neq j)$, $i = 1,2,\ldots$ *such that* (2.31) *holds, then*

$$\operatorname{sign} g_1^{[j]}(x) = +1, \quad j = 0,1,\ldots. \tag{2.34}$$

Proof. Let $j = 1$ and $C_i^{[1]} \equiv 1$, $i = 1,2,\ldots$. Using Definition 2 for $g_i^{[1]}$, we have

$$g_i^{[1]} = x^{\alpha_{i+1}-\alpha_1-1}(-\ln x)^{\beta_{i+1}-\beta_1-1}[(\alpha_{i+1} - \alpha_1)(-\ln x) + (\beta_{i+1} - \beta_1)x]C_i^{[1]}. \tag{2.35}$$

As $\lim_{x\to 0^+}(\beta_{i+1} - \beta_1)x = 0$ and $\lim_{x\to 0^+}(-\ln x) = +\infty$ then $\operatorname{sign} g_i^{[1]} = \operatorname{sign}(\alpha_{i+1} - \alpha_1)$, which equals (2.32) when $i = 1$.

For $j = 2$ and $i = 1,2,\ldots$ by definition we have

$$g_i^{[2]} = x^{\alpha_{i+2}-\alpha_2-1}\cdot(-\ln x)^{\beta_{i+2}-\beta_2-1}\Big\{[(\alpha_{i+2} - \alpha_2)(-\ln x) + (\beta_{i+2} - \beta_2)x]C_i^{[2]} + (-\ln x)x\frac{\mathrm{d}}{\mathrm{d}x}(C_i^{[2]})\Big\}, \tag{2.36}$$

where

$$C_i^{[2]} = \frac{(\alpha_{i+2} - \alpha_1)(-\ln x) + (\beta_{i+2} - \beta_1)x}{(\alpha_2 - \alpha_1)(-\ln x) + (\beta_2 - \beta_1)x}. \tag{2.37}$$

Known that $|(C_i^{[2]})'| < +\infty$ by the Lemma 4, as $\lim_{x\to 0^+} x(-\ln x) = 0$ we conclude immediately from (2.36) and (2.37) that

$$\operatorname{sign} g_i^{[2]} = \operatorname{sign}(\alpha_{i+2} - \alpha_2)\operatorname{sign}\frac{\alpha_{i+2} - \alpha_1}{\alpha_2 - \alpha_1}, \quad i = 1,2,\ldots. \tag{2.38}$$

Thus, from (2.38) we have

$$\operatorname{sign} g_i^{[2]} = \operatorname{sign}\frac{(\alpha_3 - \alpha_2)(\alpha_3 - \alpha_1)}{(\alpha_2 - \alpha_1)} = \operatorname{sign}\frac{\prod_{l=1}^{2}(\alpha_3 - \alpha_l)}{(\alpha_2 - \alpha_1)}. \tag{2.39}$$

For $j = 3$

$$g_i^{[3]} = x^{\alpha_{i+3}-\alpha_3-1}(-\ln x)^{\beta_{i+3}-\beta_3-1}\Big\{[(\alpha_{i+3} - \alpha_3)(-\ln x) + (\beta_{i+3} - \beta_3)x]C_i^{[3]} + (-\ln x)x\frac{\mathrm{d}}{\mathrm{d}x}(C_i^{[3]})\Big\}, \tag{2.40}$$

where

$$C_i^{[3]} = \frac{[(\alpha_{i+3} - \alpha_2)(-\ln x) + (\beta_{i+3} - \beta_2)x]C_{i+1}^{[2]} + (-\ln x)x(C_{i+1}^{[2]})'}{[(\alpha_3 - \alpha_2)(-\ln x) + (\beta_3 - \beta_2)x]C_1^{[2]} + (-\ln x)\cdot x(C_1^{[2]})'}. \tag{2.41}$$

By the Lemma 4 $|(C_i^{[3]})')| < +\infty$. Thus from (2.41) we conclude that

$$\operatorname{sign} C_i^{[3]} = \operatorname{sign} \frac{\alpha_{i+3} - \alpha_2}{\alpha_3 - \alpha_2} \operatorname{sign} \frac{C_{i+1}^{[2]}}{C_1^{[2]}} \tag{2.42}$$

and using (2.37)

$$\operatorname{sign} C_i^{[3]} = \operatorname{sign} \frac{\alpha_{i+3} - \alpha_2}{\alpha_3 - \alpha_2} \cdot \frac{\alpha_{i+3} - \alpha_1}{\alpha_3 - \alpha_1}, \quad i = 1, 2, \ldots, \tag{2.43}$$

finally from (2.40)

$$\operatorname{sign} g_i^{[3]} = \operatorname{sign} (\alpha_{i+3} - \alpha_3) \frac{\alpha_{i+3} - \alpha_2}{\alpha_3 - \alpha_2} \cdot \frac{\alpha_{i+3} - \alpha_1}{\alpha_3 - \alpha_1} \tag{2.44}$$

which is, for $i = 1$,

$$\operatorname{sign} g_1^{[3]} = \operatorname{sign} \frac{\prod_{l=1}^{3} (\alpha_4 - \alpha_l)}{\prod_{l=1}^{2} (\alpha_3 - \alpha_l)}. \tag{2.45}$$

Let us assume that for $j \geqslant 3$

$$g_i^{[j]} = x^{\alpha_{i+j} - \alpha_j - 1} \cdot (-\ln x)^{\beta_{i+j} - \beta_j - 1} \Big\{ [(\alpha_{i+j} - \alpha_j)(-\ln x) + (\beta_{i+j} - \beta_j)x] C_i^{[j]} + (-\ln x) x \frac{\mathrm{d}}{\mathrm{d}x} (C_i^{[j]}) \Big\}, \quad i = 1, 2, \ldots, \tag{2.46}$$

where

$$C_i^{[j]} = \frac{[(\alpha_{i+j} - \alpha_{j-1})(-\ln x) + (\beta_{i+j} - \beta_{j-1})x] C_{i+1}^{[j-1]} + (-\ln x) x (C_{i+1}^{[j-1]})'}{[(\alpha_j - \alpha_{j-1})(-\ln x) + (\beta_j - \beta_{j-1})x] C_1^{[j-1]} + (-\ln x) x (C_1^{[j-1]})'} \text{ with } |(C_i^{[j]})'| < \infty \tag{2.47}$$

and

$$\operatorname{sign} g_i^{[j]} = \operatorname{sign}(\alpha_{i+j} - \alpha_j) \operatorname{sign} c_i^{[j]} = \operatorname{sign} \frac{\prod_{l=1}^{j} (\alpha_{i+j} - \alpha_l)}{\prod_{l=1}^{j-1} (\alpha_j - \alpha_l)}. \tag{2.48}$$

Using Definition 2 and Eqs. (2.46)–(2.48) expression for $g_i^{[j+1]}$ is given by

$$g_i^{[j+1]} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g_{i+1}^{[j]}}{g_1^{[j]}}\right) = \frac{\mathrm{d}}{\mathrm{d}x}(x^{\alpha_{i+j+1}-\alpha_{j+1}}(-\ln x)^{\beta_{i+j+1}-\beta_{j+1}}C_i^{[j+1]})$$

$$= x^{\alpha_{i+j+1}-\alpha_{j+1}-1}(-\ln x)^{\beta_{i+j+1}-\beta_{j+1}-1}\Big\{[(\alpha_{i+j+1}-\alpha_{j+1})(-\ln x)+(\beta_{i+j+1}-\beta_{j+1})x]C_i^{[j+1]}$$

$$+(-\ln x)x\frac{\mathrm{d}}{\mathrm{d}x}(C_i^{[j+1]})\Big\}, \tag{2.49}$$

where

$$C_i^{[j+1]} = \frac{[(\alpha_{i+j+1}-\alpha_j)(-\ln x)+(\beta_{i+j+1}-\beta_j)x]C_{i+1}^{[j]}+(-\ln x)x(C_{i+1}^{[j]})'}{[(\alpha_{j+1}-\alpha_j)(-\ln x)+(\beta_{j+1}-\beta_j)x]C_1^{[j]}+(-\ln x)x(C_1^{[j]})'}. \tag{2.50}$$

So, (2.46) and (2.47) hold for every $j \geqslant 3$. Besides, for sufficiently small x

$$\operatorname{sign} g_i^{[j+1]} = \operatorname{sign}(\alpha_{i+j+1}-\alpha_{j+1})\operatorname{sign} C_i^{[j+1]} \tag{2.51}$$

and using (2.50) and (2.47)

$$\operatorname{sign} C_i^{[j+1]} = \operatorname{sign}\frac{(\alpha_{i+j+1}-\alpha_j)}{(\alpha_{j+1}-\alpha_j)}\operatorname{sign}\frac{C_{i+1}^{[j]}}{C_1^{[j]}}$$

$$= \operatorname{sign}\frac{(\alpha_{i+j+1}-\alpha_j)}{(\alpha_{j+1}-\alpha_j)}\operatorname{sign}\frac{(\alpha_{i+j+1}-\alpha_{j-1})}{(\alpha_{j+1}-\alpha_{j-1})}\frac{C_{i+2}^{[j-1]}}{C_2^{[j-1]}} \tag{2.52}$$

by several repetitions of (2.47)

$$\operatorname{sign} g_i^{[j+1]} = \operatorname{sign}\frac{\prod_{l=1}^{j+1}(\alpha_{i+j+1}-\alpha_l)}{\prod_{l=1}^{j}(\alpha_{j+1}-\alpha_l)}. \tag{2.53}$$

Thus (2.48) is valid for any $j \geqslant 1$. In particular (2.33) holds if we make $i = 1$ in (2.48). □

In case (iv) (see (2.11)) the study of the sign of $g_i^{[j]}(x)$, $j = 1, 2, \ldots$ can involve hard computational work giving rise to quite complicated formulas. Here, we will analyse the particular case (where the α's and β's are a priori known) occurring in some applications, as for example [7, Example 2]:

$$g_i(x) = \begin{cases} x^{(i+1)/2}(-\ln x), & i \text{ odd } (\alpha_i \equiv i,\ \beta_i \equiv 1) \\ x^{i/2}, & i \text{ even } (\alpha_i \equiv i,\ \beta_i \equiv 0), \qquad i = 1, 2, \ldots, \end{cases} \tag{2.54}$$

For this example, even for relatively low values of j the computation of $g_i^{[j]}(x)$ is quite long and tedious. This justifies the use of a symbolic programming language such as "Mathematica" [9], which gives the following expressions:

$$g_1^{[0]}(x) = g_1(x) = x(-\ln x) > 0 \quad \forall\, x \in (0,1), \tag{2.55}$$

$$g_1^{[1]}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g_2^{[0]}(x)}{g_1^{[0]}(x)}\right) = \frac{1}{x(-\ln x)^2} > 0 \quad \forall\, x \in (0,1), \tag{2.56}$$

$$g_1^{[2]}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g_2^{[1]}(x)}{g_1^{[1]}(x)}\right) = (-\ln x)(-2 + (-\ln x)) > 0 \quad \forall\, x \in (0,\mathrm{e}^{-2}), \tag{2.57}$$

$$g_1^{[3]}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g_2^{[2]}(x)}{g_1^{[2]}(x)}\right) = \frac{1}{x(-2 + (-\ln x))^2} > 0 \quad \forall\, x \in (0,1), \tag{2.58}$$

$$g_1^{[4]}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g_2^{[3]}(x)}{g_1^{[3]}(x)}\right) = 4(-2 + (-\ln x))(-3 + (-\ln x)) > 0 \quad \forall\, x \in (0,\mathrm{e}^{-3}), \tag{2.59}$$

$$g_1^{[5]}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g_2^{[4]}(x)}{g_1^{[4]}(x)}\right) = \frac{1}{x(-3 + (-\ln x))^2} > 0 \quad \forall\, x \in (0,1), \tag{2.60}$$

$$g_1^{[6]}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g_2^{[5]}(x)}{g_1^{[5]}(x)}\right) = 3(-11 + 3(-\ln x))(-3 + (-\ln x)) > 0 \quad \forall\, x \in (0,\mathrm{e}^{-11/3}); \tag{2.61}$$

thus

$$\lim_{x\to 0^+} g_1^{[j]} = \infty \quad j = 0,1,\ldots,6. \tag{2.62}$$

From the expressions (2.55)–(2.61) we see that for the first positive even integers j, $g_1^{[j]}$ is a quadratic polynomial in $(-\ln x)$ with integer coefficients, like $a_j(-\ln x)^2 + b_j(-\ln x) + c_j$, $a_j > 0$, and for j odd a rational function of the form $1/[x(\tilde{a}_j(-\ln x)^2 + \tilde{b}_j(-\ln x) + \tilde{c}_j)]$ $\tilde{a}_j > 0$. In both cases (2.62) holds and therefore $\operatorname{sign} g_1^{[j]} = +1$ $j = 0,1,\ldots,6$ for sufficiently small $x > 0$. Although it seems there is a recurrent pattern for the expressions of the $g_1^{[j]}$'s we have not found neither an indirect and nor a straightforward way of proving that $\operatorname{sign} g_1^{[j]} = +1$ for all $j \in \mathbb{N}$. Nevertheless in practice, to implement the E-algorithm only a finite number of g_i's are needed. For instance, in [7, Example 2] only the first 12 g_i's were used and in what concerns these ones the respective (2.55)–(2.61) expressions and the remaining we have also computed give that the $g_1^{[j]}$'s are positive for *sufficiently small* $x \in (0,1)$.

Remark 8. It is quite remarkable how easy becomes the determination of the sign of $g_1^{[j]}$ if we associate two consecutive g_i's as follows. For instance, suppose that $g_i(x) = x^i(-\ln x)$ and $g_{i+1}(x) = x^i$, consider

$$g_i^{[0]}(x) = g_i(x) + g_{i+1}(x) = (x^i(-\ln x) + x^i) = x^i((-\ln x) + 1). \tag{2.63}$$

Then,

$$g_i^{[1]}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g_{i+1}^{[0]}}{g_1^{[0]}}(x)\right) = \frac{\mathrm{d}}{\mathrm{d}x}(x^i) = ix^{i-1}, \quad i = 1,2,\ldots. \tag{2.64}$$

So

$$g_1^{[1]}(x) = 1,$$

$$g_i^{[2]}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g_{i+1}^{[1]}}{g_1^{[1]}}(x)\right) = \frac{\mathrm{d}}{\mathrm{d}x}((i+1)x^i) = (i+1)ix^{i-1}, \quad i = 1,2,\ldots. \tag{2.65}$$

So

$$g_1^{[2]}(x) = 2$$

and

$$g_1^{[j]} = j, \quad j = 1,2,\ldots. \tag{2.66}$$

The same type of simplification occurs in the following example given in [7, Example 4] where we can associate four consecutive g_i's and obtain:

$$g_i^{[0]}(x) = x^i(-\ln x)^3 + x^i(-\ln x)^2 + x^i(-\ln x) + x^i \quad i = 1,2,\ldots. \tag{2.67}$$

Then $\forall\, x \in (0,1)$ $\operatorname{sign} g_1^{[j]}(x) = +1$ since

$$g_1^{[j]}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g_2^{[j-1]}}{g_1^{[j-1]}}(x)\right) = j, \quad j = 1,2,\ldots. \tag{2.68}$$

From a theorem of Brezinski (see [3, Theorem 2.6, p. 66]), we know that the extrapolated value $E_k^{(n)}$ obtained by application of the E-algorithm using k auxiliary sequences $(g_1(n)),\ldots,(g_k(n))$ is the same as the one obtained if one or several of such g's are replaced by linear combinations of the others. We think that this result, and the association of consecutive g_i's as described above, could be useful in the study of the sign of $g_1^{[j]}$. These questions deserve further studies.

3. Sign regularity

We know that if $f'(x) > 0$ then $x_1 < x_2 \Rightarrow f(x_2) - f(x_1) > 0$. A generalization of this idea is given by the following theorem where the positivity of a certain determinant results from the positivity of several Wronskians.

The Wronskians, $W(g_1), W(g_1,g_2),\ldots,W(g_1,g_2,\ldots,g_m)$, acting on a set of m real functions

$$g_i(x), \quad i = 1,2,\ldots,m, \tag{3.1}$$

are defined by

$$W(g_1,g_2,\ldots,g_m)(x) = \begin{vmatrix} g_1(x) & \frac{\mathrm{d}}{\mathrm{d}x}g_1(x) & \cdots & \frac{\mathrm{d}^{(m-1)}g_1(x)}{\mathrm{d}x^{m-1}} \\ \vdots & \vdots & \vdots & \\ g_m(x) & \frac{\mathrm{d}}{\mathrm{d}x}g_m(x) & \cdots & \frac{\mathrm{d}^{(m-1)}g_m(x)}{\mathrm{d}x^{m-1}} \end{vmatrix}. \tag{3.2}$$

Theorem 9 (Karlin [5, Theorem 2.3, p. 52]). *Let* $g_1(x), g_2(x), \ldots, g_m(x) \in C^{m-1}(a,b)$ *and* ε_i, $i = 1,2,\ldots$, *be a sequence of numbers either* $+1$ *or* -1 (*i.e. the sequence* (ε_i) *of signs may or may not follow a regular pattern*).

If $\varepsilon_1 W(g_1) > 0, \varepsilon_2 W(g_1, g_2) > 0, \ldots, \varepsilon_m(g_1, g_2, \ldots, g_m) > 0$ *then*

$$\forall x_i \in (a,b){:}\ x_1 \leqslant x_2 \leqslant x_3 \leqslant \cdots \leqslant x_m,$$

$$\varepsilon_m \begin{vmatrix} g_1(x_1) & g_1(x_2) & \cdots & g_1(x_m) \\ g_2(x_1) & g_2(x_2) & \cdots & g_2(x_m) \\ \vdots & \vdots & & \vdots \\ g_m(x_1) & g_m(x_2) & \cdots & g_m(x_m) \end{vmatrix} > 0. \tag{3.3}$$

In order to analyse the signs of the Wronskians, if the functions $g_i(x)$ defined as in (2.7) are used, we need the following

Lemma 10. *Let us consider the m functions in* $C^{m-1}(0,1)$ $g_i(x) = g_i^{[0]}(x)$, $i = 1,2,\ldots$ *and define* $g_1^{[j]}(x)$ *as in* (2.12). *Then*

$$W(g_1, g_2, \ldots, g_m)(x) = (g_1(x))^m (g_1^{[1]}(x))^{m-1} (g_1^{[2]}(x))^{m-2} \cdots (g_1^{[m-1]}(x)). \tag{3.4}$$

Proof. The following identity due to Karlin [5, Remark (v), p. 53] enables us to reduce the order of a Wronskian by one:

$$W(g_1, g_2, \ldots, g_m)(x) = (g_1)^m W(g_1^{[1]}, g_2^{[1]}, \ldots, g_{m-1}^{[1]}), \tag{3.5}$$

where

$$g_1^{[1]} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g_2}{g_1}\right), \quad g_2^{[1]} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g_3}{g_1}\right), \ldots, g_{m-1}^{[1]} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g_m}{g_1}\right).$$

We will use induction on m to prove (3.4). For $m = 1$ the relation is trivially true. Let $m = 2$, using (3.5) we obtain

$$W(g_1, g_2)(x) = (g_1)^2 W(g_1^{[1]}) = (g_1)^2 g_1^{[1]}. \tag{3.6}$$

Suppose now that for $k \geqslant 2$

$$W(g_1, g_2, \ldots, g_k) = g_1^k (g_1^{[1]})^{k-1} (g_1^{[2]})^{k-2} \cdots (g_1^{k-1}), \tag{3.7}$$

where

$$g_1^{[1]} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g_2}{g_1}(x)\right), \quad g_1^{[2]} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g_2^{[1]}}{g_1^{[1]}}\right), \quad \ldots, \quad g_1^{[k-1]} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g_2^{[k-2]}}{g_1^{[k-2]}}\right).$$

From the identity (3.5) we have

$$W(g_1, g_2, \ldots, g_k, g_{k+1}) = (g_1)^{k+1} W(g_1^{[1]}, g_2^{[1]}, g_3^{[1]}, \ldots, g_k^{[1]}) \tag{3.8}$$

and we can now use the induction hypothesis (3.7) to obtain

$$W(g_1^{[1]}, g_2^{[1]}, g_3^{[1]}, \ldots, g_k^{[1]}) = (g_1^{[1]})^k [(g_1^{[1]})^{[1]}]^{k-1} [(g_1^{[1]})^{[2]}]^{k-2} \cdots (g_1^{[1]})^{[k-1]}. \tag{3.9}$$

But by definition

$$g_1^{[2]} = (g_1^{[1]})^{[1]} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g_2^{[1]}}{g_1^{[1]}}\right) \qquad g_1^{[3]} = (g_1^{[1]})^{[2]}, \quad \ldots, \quad g_1^{[k]} = (g_1^{[1]})^{[k-1]} \tag{3.10}$$

so, (3.8) becomes

$$W(g_1, g_2, \ldots, g_{k+1}) = g_1^{k+1} (g_1^{[1]})^k (g_1^{[2]})^{k-1} \cdots (g_1^{[k]}) \tag{3.11}$$

and therefore (3.4) holds for any $m \geqslant 1$. $\square$

Corollary 11. *Let* (i) $g_i(x) = g_i^{[0]}(x) = x^\alpha(-\ln x)^{\beta_i}$, $\alpha \in \mathbb{R}^+$, $\beta_1 > \beta_2 > \cdots > \beta_m > \cdots > 0$ *or*
(ii) $g_i(x) = x^{\alpha_i}(-\ln x)^{\beta_i}$, $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < \cdots$, $\beta_1 > \beta_2 > \cdots > \beta_m > \cdots > 0$ *such that* (2.31) *holds or*
(iii) $g_i(x) = x^{\alpha_i}$, $0 < \alpha_1 < \cdots < \alpha_m < \cdots$ *or* $g_i(x) = (-\ln x)^{\beta_i}$, $\beta_1 > \beta_2 > \cdots > 0$, *then in case* (i) *or* (iii) $\forall x_i$: $x_1 \leqslant x_2 \leqslant x_3 \leqslant \cdots \leqslant x_m$ *in* (a,b) *in case* (ii) $\forall$ *sufficiently small* x_i: $x_1 \leqslant x_2 \leqslant x_3 \leqslant \cdots \leqslant x_m$ *in* (a,b)

$$\begin{vmatrix} g_1(x_1) & g_1(x_2) & \cdots & g_1(x_m) \\ g_2(x_1) & g_2(x_2) & \cdots & g_2(x_m) \\ \vdots & \vdots & & \vdots \\ g_m(x_1) & g_m(x_2) & \cdots & g_m(x_m) \end{vmatrix} > 0. \tag{3.12}$$

Proof. In case (i) or (iii) by Lemma 3 we can say that $g_1^{[j]}(x) > 0$, $\forall j \in \mathbb{N}$, $\forall x \in (0,1)$ and then using (3.4) $W(g_1, g_2, \ldots, g_m) > 0$ $\forall m \geqslant 1$. Thus, by the Theorem 9 we conclude (3.12). The same arguments are valid in the case (ii) using the Corollary of Lemma 6. $\square$

Comparing (3.12) with the determinant in Theorem 1, we need to consider the effect on the respective sign when we reorder $g_1, g_2, \ldots, g_m$ into $g_m, \ldots, g_2, g_1$:

Let A be a matrix of order p and $|A|$ its determinant. If we completely reverse the order of the rows (or columns) of A the value of the resulting determinant is given by $(-1)^{p(p-1)/2}|A|$. Using transposition and this last remark we can write

$$\begin{vmatrix} g_1(x_1) & g_1(x_2) & \cdots & g_1(x_m) \\ g_2(x_1) & g_2(x_2) & \cdots & g_2(x_m) \\ \vdots & \vdots & & \vdots \\ g_m(x_1) & g_m(x_2) & \cdots & g_m(x_m) \end{vmatrix} = \begin{vmatrix} g_1(x_1) & g_2(x_1) & \cdots & g_m(x_1) \\ g_1(x_2) & g_2(x_2) & \cdots & g_m(x_2) \\ \vdots & \vdots & & \vdots \\ g_1(x_m) & g_2(x_m) & \cdots & g_m(x_m) \end{vmatrix}$$

$$= (-1)\frac{m(m-1)}{2} \begin{vmatrix} g_m(x_1) & g_{m-1}(x_1) & \cdots & g_1(x_1) \\ g_m(x_2) & g_{m-1}(x_2) & \cdots & g_1(x_2) \\ \vdots & \vdots & & \vdots \\ g_m(x_m) & g_{m-1}(x_m) & \cdots & g_1(x_m) \end{vmatrix}. \tag{3.13}$$

So, $\forall\, m \geqslant 1$, $\forall\, x_1 \leqslant x_2 \leqslant \cdots \leqslant x_m$, the inequality (3.12) becomes

$$(-1)\frac{m(m-1)}{2} \begin{vmatrix} g_m(x_1) & g_{m-1}(x_1) & \cdots & g_1(x_1) \\ g_m(x_2) & g_{m-1}(x_2) & \cdots & g_1(x_2) \\ \vdots & \vdots & & \vdots \\ g_m(x_m) & g_{m-1}(x_m) & \cdots & g_1(x_m) \end{vmatrix} > 0 \tag{3.14}$$

and the relation (3.14) means that for $X = (0,1)$ and $Y = \mathbb{N}$ the two--variable function $K(x,y) = g_y(x)$ is *sign regular*, i.e.,

Definition 12 (Karlin [5, pp. 11–12]). A real function $K(x,y)$ ranging over ordered sets X and Y, respectively, is said to be *sign regular* if

$$\forall\, x_i \in X,\ \forall\, y_j \in Y\colon\ x_1 \leqslant x_2 \leqslant \cdots \leqslant x_m,\ y_1 \leqslant y_2 \leqslant \cdots \leqslant y_m \quad \text{and} \quad \forall\, m \in \mathbb{N}$$

there exists a sequence of numbers ε_m each either $+1$ or -1 such that

$$\varepsilon_m K\begin{pmatrix} x_1, & x_2, & \ldots, & x_m \\ y_1, & y_2, & \ldots, & y_m \end{pmatrix} = \varepsilon_m \begin{vmatrix} K(x_1,y_1) & K(x_1,y_2) & \cdots & K(x_1,y_m) \\ K(x_2,y_1) & K(x_2,y_2) & \cdots & K(x_2,y_m) \\ \vdots & \vdots & & \vdots \\ K(x_m,y_1) & K(x_m,y_2) & \cdots & K(x_m,y_m) \end{vmatrix} \geqslant 0. \tag{3.15}$$

The remainder of this section is dedicated to show that if we consider decreasing $x_i \in (a,b)$ the positivity of the determinant in (3.12) is maintained. This is afterwards used to conclude on the validity of Theorem 1 when applied to the auxiliary sequences we are concerned with.

In order to achieve the conclusion that the family $g_i(x) = x^{\alpha_i}(-\ln x)^{\beta_i}$, defined in (2.7), does verify the determinantal inequality in Theorem 1 note that the real variable x (usually representing the step size h) is, in the applications, monotically related to the integer variable n, for instance via the strictly decreasing function $n \mapsto x = 1/2^{n+1}$. So, in practice, we define a sequence (x_n) of positive numbers tending to 0 as n tends to ∞, thus $\exists N$ such that $\forall n \geqslant N$, x_n is *sufficiently small* in $(a,b) = (0,1)$.

The next theorem establishes that the sign regularity of $K(x,y)$ is preserved if we transform one or both the independent variables into new variables by means of strictly monotone functions:

Theorem 13 (Karlin [5, Theorem 2.1, p. 18]). *Let $K(x,y)$ be a sign regular function ($x \in X$, $y \in Y$) and let $u = \phi^{-1}(x)$, $v = \psi^{-1}(y)$ (ϕ^{-1} and ψ^{-1} are the inverse functions of ϕ and ψ respectively). Consider*

$$L(u,v) = K(\phi(u), \psi(v)], \quad u \in U, \; v \in V, \tag{3.16}$$

where ϕ and ψ are strictly monotone functions mapping U, V onto X, Y. Then

(i) *If ϕ and ψ are both strictly increasing functions, $L(u,v)$ is sign regular and*

$$\varepsilon_m(K) = \varepsilon_m(L), \quad m = 1,2,\ldots \tag{3.17}$$

($\varepsilon_m(K)$ denotes the sign ε_m of K).

(ii) *If ϕ (or ψ) is strictly increasing while ψ (or ϕ) is strictly decreasing, then $L(u,v)$ is sign regular and*

$$\varepsilon_m(K) = (-1)^{m(m-1)/2} \varepsilon_m(L). \tag{3.18}$$

So, if to the variable $n \in \mathbb{N}$: $n < n+1 < \cdots$ corresponds $x_n \in (0,1)$: $x_n > x_{n+1} > x_{n+2} > \cdots$ (as for instance $x_n = 1/2^{n+1}$, $n = 0,1,\ldots$) then by (ii) above we can write

$$\begin{vmatrix} g_m(n) & g_{m-1}(n) & \cdots & g_1(n) \\ g_m(n+1) & g_{m-1}(n+1) & \cdots & g_1(n+1) \\ \vdots & \vdots & & \vdots \\ g_m(n+m-1) & g_{m-1}(n+m-1) & \cdots & g_1(n+m-1) \end{vmatrix}$$

$$= (-1)\frac{m(m-1)}{2} \begin{vmatrix} g_m(x_n) & g_{m-1}(x_n) & \cdots & g_1(x_n) \\ g_m(x_{n+1}) & g_{m-1}(x_{n+1}) & \cdots & g_1(x_{n+1}) \\ \vdots & \vdots & & \vdots \\ g_m(x_{n+m-1}) & g_{m-1}(x_{n+m-1}) & \cdots & g_1(x_{n+m-1}) \end{vmatrix} > 0, \tag{3.19}$$

where the last inequality is a consequence of (3.14). Comparing (2.2) and (3.19) we can conclude that the determinantal inequality in Theorem 1 is satisfied by the family of auxiliary functions which we are dealing with. This means that we can be sure that the E-algorithm accelerates the convergence of the sequences whose truncation error has g_i's of the form given in (2.8)–(2.11). And so the corresponding sequences (S_n): $S_n - S = a_1 g_1(x_n) + a_2 g_2(x_n) + \cdots$ can also be accelerated by the E-algorithm.

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